Nithyanandan Thyagarajan (CSIRO)
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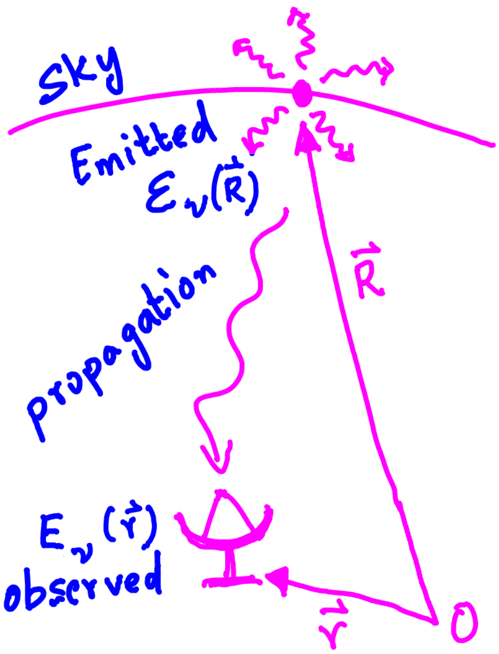
Fourier Transforms in interferometry



Emitted E-field:

$$\mathcal{E}_\nu(\vec{R}) \equiv \mathcal{E}_\nu(\hat{s})$$

Received E-field: $E_\nu(\vec{r})$

Assumptions

- A1 : Scalar (no polarization)
- A2 : Emission from surface at a long distance
- A3 : No source inside of the celestial sphere

$$\boxed{E_\nu(\vec{r}) = \int_S \mathcal{E}_\nu(\vec{R}) \frac{e^{i\frac{2\pi}{\lambda}|\vec{R}-\vec{r}|}}{|\vec{R}-\vec{r}|} \, ds}$$

Spatial coherence measured between two antennas at $\vec{r}_1, \vec{r}_2$: $V_\nu(\vec{r}_1,\vec{r}_2) = \langle E_\nu(\vec{r}_1) E_\nu^*(\vec{r}_2)\rangle$

A4: Incoherent emission and spatial stationarity

$$\Rightarrow \langle \mathcal{E}_\nu(\vec{R}_1) \mathcal{E}_\nu^*(\vec{R}_2)\rangle = \begin{cases} \langle |\mathcal{E}_\nu(R_1)|^2\rangle, & \vec{R}_1 = \vec{R}_2 \\ 0, & \vec{R}_1 \neq \vec{R}_2 \end{cases}$$

$$\Rightarrow V_\nu(\vec{r}_1,\vec{r}_2) \equiv V_\nu(\vec{r}_1 - \vec{r}_2)$$

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$$\boxed{\begin{aligned} V_\nu(\vec{r}_1, \vec{r}_2) &= \langle E_\nu(\vec{r}_1)\, E_\nu^*(\vec{r}_2)\rangle \\ &\approx \int_S \langle |\mathcal{E}_\nu(\hat{s})|^2\rangle\, e^{-i\frac{2\pi}{\lambda}\hat{s}\cdot(\vec{r}_1-\vec{r}_2)}\, d\Omega \end{aligned}}$$

The spatial coherence function, $V_\nu(\vec{r}_1, \vec{r}_2)$ is only a function of $\vec{r}_1 - \vec{r}_2$, which is referred to as spatial stationarity.

Define the coordinates in terms of their components,

$$\hat{s} \equiv (\ell, m, n), \quad \vec{r}_1 - \vec{r}_2 \equiv (\lambda u, \lambda v, \lambda w)$$

where, ℓ, m, and n are the direction cosines,

$$\ell^2 + m^2 + n^2 = 1 \Rightarrow n = \sqrt{1-\ell^2-m^2}$$

Then,

$$V_\nu'(u,v,w) = \iint I_\nu(\ell,m)\, \frac{e^{-i2\pi[u\ell+vm+wn]}}{\sqrt{1-\ell^2-m^2}}\, d\ell\, dm$$

$$= \iint I_\nu(\ell,m)\, \frac{e^{-i2\pi[u\ell+vm+w\sqrt{1-\ell^2-m^2}]}}{\sqrt{1-\ell^2-m^2}}\, d\ell\, dm$$

$$= e^{-i2\pi w} \iint I_\nu(\ell,m)\, \frac{e^{-i2\pi[u\ell+vm+w(\sqrt{1-\ell^2-m^2}-1)]}}{\sqrt{1-\ell^2-m^2}}\, d\ell\, dm$$

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The spatial coherence (or visibility) with respect to a phase center, $\hat{s}_0 \equiv (0,0,1)$ is given by

$$\boxed{\begin{aligned} V_\nu(u,v,w) &= e^{i2\pi w}\, V_\nu'(u,v,w) \\ &= \iint I_\nu(l,m)\, \frac{e^{-i2\pi[ul+vm+w(\sqrt{1-l^2-m^2}-1)]}}{\sqrt{1-l^2-m^2}}\, dl\, dm \end{aligned}}$$

The quantity, $w(\sqrt{1-l^2-m^2}-1)$ is the phase term determines if the above equation can be reduced to a Fourier transform. It is possible when

$$\boxed{w\left(\sqrt{1-l^2-m^2}-1\right) \approx 0}$$

A5: Measurements on a plane (co-planarity) $w=0$

A6: Emission from small region of sky $n=\sqrt{1-l^2-m^2} \simeq 1$

Under A5 (co-planarity), $w=0$

$$V_\nu(u,v,w\equiv 0) = \iint I_\nu(l,m)\, \frac{e^{-i2\pi(ul+vm)}}{\sqrt{1-l^2-m^2}}\, dl\, dm$$

$$\boxed{V_\nu(u,v,w\equiv 0) \xleftrightarrow{\text{F.T}} \frac{I_\nu(l,m)}{\sqrt{1-l^2-m^2}}}$$

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Under A6 (emission from small region of sky),

$$\sqrt{1-\ell^2-m^2} \approx 1$$

$$\Rightarrow \boxed{\begin{aligned} V_\nu(u,v,w) &\equiv V_\nu(u,v) \\ &= \iint I_\nu(\ell,m)\, e^{-i2\pi(u\ell+vm)}\, d\ell\, dm \end{aligned}}$$

Note that $V_\nu(u,v,w) \equiv V_\nu(u,v)$ is independent of any w-term. And,

$$\boxed{V_\nu(u,v) \xleftrightarrow{F.T} I_\nu(\ell,m)}$$

In summary so far,

$$\boxed{V_\nu(u,v,w) = \iint I_\nu(\ell,m)\, \frac{e^{-i2\pi[u\ell+vm+\underline{w(\sqrt{1-\ell^2-m^2}-1)}]}}{\sqrt{1-\ell^2-m^2}}\, d\ell\, dm}$$

A5 $w=0$

A6 $n \approx 1$

$$\boxed{V_\nu(u,v,w\equiv 0) \xleftrightarrow{F.T} \frac{I_\nu(\ell,m)}{\sqrt{1-\ell^2-m^2}}} \qquad \boxed{V_\nu(u,v) \xleftrightarrow{F.T} I_\nu(\ell,m)}$$

What happens when $w(\sqrt{1-\ell^2-m^2}-1) \not\approx 0$?

→ "w-projection" by Cornwell, Golap & Bhatnagar (2008)

→ "w-stacking" by Offringa et al. (2014)

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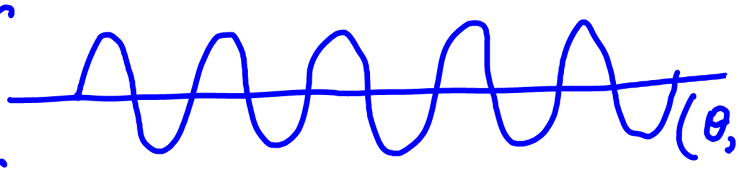
An interesting example occurs when making wide-field measurements, that is, the assumption, **A6**, is not valid.

Usually, interferometers do not measure the auto-correlations, or $\vec{r}_1 = \vec{r}_2$. A statement that is commonly made is that because the interferometer does not measure zero-spacing, it does not measure the monopole or the sky-averaged component of the intensity distribution on the sky. That is, because

$$V_\nu(0,0,0) = 0 \Rightarrow \langle I_\nu(\hat{s}) \rangle = 0$$

and that $V_\nu(\Delta r_{ab} \neq 0)$ cannot provide information on the monopole component, $\langle I_\nu(\hat{s}) \rangle$.

In Fourier transform theory, it can be understood as follows. Each $\vec{r}_1 - \vec{r}_2 \equiv (\lambda u, \lambda v, \lambda w)$ corresponds to a sinusoidal fringe on the sky. Over a sufficient area, the sine/cosine functions average to zero upon integration over the full field of view.

$\int_S$  $(\theta, \phi) = 0$

However, that is valid only when $l^2 + m^2 \ll 1$.

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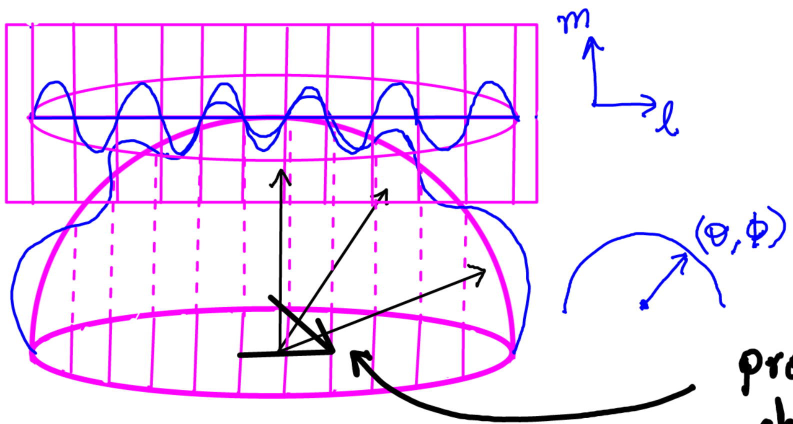
The (l, m, n) coordinates describe locations on the tangential plane of the celestial sphere. When $l^2 + m^2 << 1$ it implies that the tangential plane is very close and approximates the celestial sphere in that direction. When $l^2 + m^2 \not\ll 1$, the tangent plane deviates significantly from the celestial sphere.



Uniform interval in (l, m) gives non-uniform intervals in (θ, ϕ) because the projected baseline length changes with zenith angle.

$$\int e^{-i2\pi(ul+vm)}\, dl\, dm = 0$$

But, $$\int e^{-i2\pi(ul+vm)}\, d\Omega = \int \frac{e^{-i2\pi(ul+vm)}}{\sqrt{1-l^2-m^2}}\, dl\, dm \neq 0$$

Another reasoning:

→ Baseline foreshortening (projected baseline is direction-dependent)

→ Sensitive to small scales near zenith, but become sensitive to large scales at large zenith angles

→ Extreme case: near horizon (ZA → 90°), becomes sensitive to the monopole (sky-averaged) signal.

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This phenomenon is called the "pitchfork" effect and is described in detail in Thyagarajan et al. (2015a)

This is also the reason that the PSF is not shift-invariant in wide-field imaging. That is, the PSF is direction-dependent. Due to the foreshortening, the PSF in angular coordinates is narrower close to zenith and wider at wide zenith angles. Deconvolution will require using a direction-dependent PSF.